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Transitivity and Ergodicity of Quantum Systems

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We generalize the notion of a topological transitive or a topologically mixing system for quantum mechanical systems in a consistent way. We compare these ergodic properties with the classical results. We deal with some aspects of nearly Abelian systems and investigate some relations between these notions.

KEY WORDS: Ergodicity; mixing; asymptotic Abelianness; transitivity; extremal invariant states.

1. INTRODUCTION

Boltzmann's visions on statistical mechanics have developed into two mathematical disciplines, topological dynamics and measure-theoretic ergodic theory. They contain related but not identical information. In Boltzmann's times these notions were not sharpened to the extreme that one could say which one he had in mind. In any case, neither of the two theories contains nature, which is quantum mechanical.

In this paper we show that the algebraic formulation of quantum mechanics contains notions for ergodicity which reduce to the classical ones for an Abelian algebra and therefore generalize these theories to more realistic cases. It also clearly separates the two theories. The generalized topological dynamics is concerned with the long-time behavior of one-parameter group of isomorphisms of a C^*-algebra whereas quantum mechanical ergodic theory studies the properties of extremal invariant states, that is, invariant states which cannot be decomposed into other invariant states.

We will see that by going to the non-Abelian generalization the

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ergodic properties become partly better and partly worse. What gets lost is that the extremal invariant states automatically have the desired ergodic properties. However, for these one does not need Abelianness, but only asymptotic Abelianness, which requires that observables at vastly different times commute. This never holds for finite quantum systems, but once they are discarded, one is much closer to the classical ideas.

An ergodic (or mixing) state tells us that on its support the system approaches equilibrium in the time mean (or time limit). Thus, if there is a faithful ergodic state, the system approaches equilibrium as a whole. In fact, we will be able to draw this conclusion already if the representation furnished by this state is faithful.

Now comes a special quantum bonus, namely, if the algebra is simple, the latter condition is satisfied by necessity since simple algebras have only faithful nontrivial representations. Simple algebras are the opposite extreme to classical systems; their classical part, namely their center, is trivial.

The situation becomes even better if also the algebra which contains all weak limit elements has no center. Then the time limits exist and the system does not only show ergodic, but even mixing properties. This last condition is satisfied for extremal KMS states, which, ignoring some obnoxious examples, practically always exist.

Thus, we see that in the non-Abelian setting there is a large class of systems, namely the simple asymptotic Abelian algebras, which exhibit the features which Boltzmann envisaged.

2. GENERAL DEFINITIONS

In the sequel algebras are assumed to be with an *identity*, *normal* shall mean continuity with monotone limits, and all means are understood to be *invariant means*.

Definition 1. A *C^*-dynamical system* $(\mathscr{A}, \alpha)$ is a pair consisting of a C^*-algebra $\mathscr{A}$ and an automorphism α ($\neq$ id) of $\mathscr{A}$. The set of (α-) invariant elements of $\mathscr{A}$ is denoted by $\mathscr{I} := \{a \in \mathscr{A} : \alpha(a) = a\}$.

Remarks

1. To be precise, we just defined something one should call a *discrete* C^*-dynamical system, because the set of the automorphisms we look at form a group which is isomorphic to $\mathbb{Z}$. Of course, one can use other groups, such as $\mathbb{R}$, or even more exotic (topological) groups (ref. 1, p. 136).

2. In the case the C^*-algebra is even a W^*-algebra such a system is called a *W^*-dynamical system*. We denote it by $(\mathscr{M}, \alpha)$ to distinguish it from a common C^*-dynamical system $(\mathscr{A}, \alpha)$.

If $(\mathscr{A}, \alpha)$ has an invariant state ω, the automorphism α is unitarily represented via the GNS construction π_ω. The α can then be extended to $\pi_\omega(\mathscr{A})''$, which thus is then a W^*-dynamical system containing $(\mathscr{A}, \alpha)$. If there are several invariant states, this W^*-dynamical system clearly is not uniquely determined by the original C^*-dynamical system $(\mathscr{A}, \alpha)$.

Note that, on the other hand, it is always impossible to reconstruct a unique $\mathscr{A}$ out of some given $\mathscr{M}$.

3. If $\mathscr{A}$ is *Abelian*, it is isomorphic to $C(M)$, the continuous functions over some compact space M (ref. 6, p. 4). In this case α corresponds to a homeomorphism α_* of M: $\alpha(f)(m) = f(\alpha_*(m))$.

For Abelian W^*-algebras there is an isomorphism between $\mathscr{M}$ and $L^\infty(M, \mu)$, the essentially bounded μ-measurable functions on some localizable measure space M (ref. 6, p. 45).

[In the Abelian case the last remark corresponds to the fact that though there is no unique Borel measure, there is always a unique (Borel-) measurable structure for some given toplogical space, but that in general it is impossible to reverse such a construction.]

4. The other extreme is $\mathscr{A}$ being a *simple* algebra, i.e., there are no nontrivial (closed) two-sided ideals. This is realized, e.g., by algebras of creation and annihilation operators in quantum field theory. Simple algebras have trivial center; a nontrivial center contains a nontrivial element which generates a (nontrivial) two-sided ideal. The converse is not true, $\mathscr{B}(\mathscr{H})$, dim $\mathscr{H} = \infty$, has trivial center, but contains the nondenumerable family of ideals $\mathscr{C}_p$, $1 \leqslant p \leqslant \infty$ [p-norm-closure of the finite-rank operators, C_∞ being closed in the C^*-topology of $\mathscr{B}(\mathscr{H})$] (ref. 5, p. 41).

5. The typical algebras we have in mind are *UHF-algebras* (or *Glimm algebras*), which are the norm closure of an ascending sequence of finite-dimensional full matrix algebras with the same identity $\mathscr{M}_d$, $d \to \infty$. In physics they correspond to systems with infrared and ultraviolet cutoffs, which are then removed. For the needs of physicists they should provide a sufficient general framework. They have the virtue that they are simple, separable, and have a unique tracial state (ref. 4, p. 205).

Definition 2. A state ω over a C^*-dynamical system $(\mathscr{A}, \alpha)$ is called *invariant* iff $\omega \circ \alpha = \omega$. The extremal points of the weak*-closed convex set $\mathscr{S}_\alpha$ of α-invariant states are called *extremal invariant.*

Remarks

1. In the Abelian (W^*-) case a state ω corresponds to a Borel measure μ on M, and S_α are the invariant measures of classical ergodic theory.

2. $\mathscr{S}_\alpha$ is nonempty, since an invariant mean over $\mathbb{Z}$ of $\omega \circ \alpha^n(\cdot)$, $n \in \mathbb{Z}$, is a positive linear functional which cannot vanish as $\omega(\alpha^n(\mathbb{1})) = 1$.

For instance, the unique tracial state τ over a UHF-algebra is always invariant, since $\tau \circ \alpha$ is also tracial,

$$\tau \circ \alpha(ab) = \tau(\alpha(\alpha)\, \alpha(b)) = \tau \circ \alpha(ba)$$

3. Any $\omega \in \mathscr{S}_\alpha$ leads to a GNS-representation π_ω of $\mathscr{A}$ in $\mathscr{B}(\mathscr{H}_\omega)$, the bounded operators over some Hilbert space $\mathscr{H}_\omega$ with cyclic vector $|\Omega\rangle \in \mathscr{H}_\omega$. Then α is implemented by some unitary element $U \in \mathscr{B}(\mathscr{H}_\omega)$:

$$\pi_\omega(\alpha(a)) = U^{-1} \pi_\omega(a)\, U$$

U can be made unique by the condition $U\,|\Omega\rangle = |\Omega\rangle$ (ref. 1, p. 42)

The invariant states may lead to unusual representations. For instance, a time mean of the Weyl operators with the free time evolution

$$e^{i(rp + sq)} \mapsto e^{i[rp + s(q + pt)]}$$

in any state of the standard representation leads to a state

$$\omega(e^{i(rp + sq)}) = \begin{cases} 0 & \text{for} \quad s \neq 0 \\ f(r) & \text{for} \quad s = 0 \end{cases}$$

Thus, the representation is not strongly continuous and acts on a non-separable Hilbert space: With $|r, s\rangle = e^{i(rp + sq)}\,|\Omega\rangle$ we have

$$\langle r', s' \,|r, s\rangle = \omega(e^{-i(r'p + s'q)} e^{i(rp + sq)}) = \begin{cases} 0 & \text{for} \quad s \neq s' \\ e^{is(r - r')/2} f(r - r') & \text{for} \quad s = s' \end{cases}$$

4. In a representation π_ω and with η_n a mean over $\mathbb{Z}$ one can define a mean η of operators by taking the mean of matrix elements

$$\omega(b\eta(a)c) = \eta_n(\omega(b\alpha^n(a)c))$$

Therefore η is a conditional expectation from $\mathscr{A}''$ onto $\bar{\mathscr{I}} := \{a \in \mathscr{A}'': \alpha(a) = a\}$, if we abbreviate $\pi_\omega(\mathscr{A})''$ by $\mathscr{A}''$, etc.

If ω is faithful and invariant, η even coincides with the canonical conditional expectation, which has the property

$$\omega = \omega\,|_{\mathscr{I}} \circ \eta$$

since $\bar{\mathscr{I}}$ is invariant under the modular automorphisms of ω, as the latter commutes with α. This is the criterion for existence and uniqueness of the canonical conditional expectation.[7] In this rather typical situation η therefore becomes independent of the particular mean chosen for η_n.

According to von Neumann's ergodic theorem, $P_0 := \eta_n(U^n)$ is the projector into the eigenspace of eigenvalue one of U and therefore is also independent of η_n. Then obviously $\eta(a)\, P_0 = P_0 \eta(a) = P_0 a P_0\ \forall a \in \pi_\omega(\mathscr{A})$.

For long times the system may behave classically in a way discussed in the following result.

Proposition 1. Among, the properties

(i) $\omega(c^*[\eta(a), b]c) = 0\ \forall a, b, c \in \mathscr{A}\ \forall \eta$ mean over $\mathbb{Z}$

(ii) $\omega([\eta(a), b]) = 0\ \forall a, b \in \mathscr{A}\ \forall \eta$ mean over $\mathbb{Z}$

(iii) $P_0 \mathscr{A} P_0$ is Abelian

(iv) $\eta(\mathscr{A}) \subset \mathscr{Z}_\omega := \mathscr{A}'' \cap \mathscr{A}'\ \forall \eta$ mean over $\mathbb{Z}$

(v) $\mathscr{A} \cap U' \subset \mathscr{Z}_\omega$

(vi) $\mathscr{A}' \cap U' \subset \mathscr{Z}_\omega$

(vii) $\mathscr{A}' \cap U'$ is Abelian

which are supposed to hold for all $\omega \in S_\alpha$, we have the following implications:

$$\begin{array}{ccccccc} \text{(iv)} & \Leftrightarrow & \text{(i)} & \Rightarrow & \text{(ii)} & \Leftrightarrow & \text{(iii)} \\ \Downarrow & & \Downarrow & & \Downarrow & & \\ \text{(v)} & & \text{(vi)} & \Rightarrow & \text{(vii)} & & \end{array}$$

Remarks

1. Property (i) is guaranteed if $\lim_{n\to\infty} \|[\alpha^n(a), b]\| = 0\ \forall a, b \in \mathscr{A}$. However, this condition may fail even if (i) holds. For instance, for Fermi fields and translations the anticommutator rather than the commutator vanishes asymptotically. However, since η maps the odd elements into zero, (i) holds nevertheless.

2. Property (i) will not necessarily hold for all $a \in \mathscr{A}''$ even if it is satisfied for all $a \in \mathscr{A}$. For instance, in an irreducible representation the unitary operators which generate a non-Abelian group commuting with α will be in $\mathscr{A}''$ but form a noncommutative subalgebra invariant under α. In general also $\mathscr{A}'' \cap U' \not\subset \mathscr{Z}_\omega$, since the generator of α may be in $\mathscr{A}''$.

3. Property (ii) (*G-Abelianness*) is satisfied for finite systems iff U is nondegenerate. In this case the sharper version (i) (*η-Abelianness*) fails and so (i) seems more appropriate to characterize infinite systems with thermodynamic behavior.

4. Note that $\mathscr{Z}_\omega$ is the center of $\mathscr{A}''$. It may be nontrivial even if $\mathscr{A}$ is simple and therefore must have a trivial center.

5. For further notions of "asymptotic Abelianness," examples, and implications see ref. 2.

Proof. (i) $\Rightarrow$ (iv): By polarization, (i) requires any matrix element of $[\eta(a), b]$ to vanish.

(iv) $\Rightarrow$ (i): This is obvious.

(iv) $\Rightarrow$ (v): η leaves the invariant elements unchanged.

(i) $\Rightarrow$ (ii): Take $c = 1$.

(ii) $\Leftrightarrow$ (iii): $\eta(\omega([\alpha^n(a), b])) = \omega([P_0 a P_0, P_0 b P_0])$; thus (iii) $\Rightarrow$ (ii). But since (ii) is asserted to hold for any expectation value in the range of P_0, we also have (ii) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (vii): Consider the C^*-algebra $\mathscr{R}$ generated by $\pi_\omega(\mathscr{A})$ and U and the algebra $\mathscr{R}' := \mathscr{A}' \cap U'$ (ref. 8, p. 134). Since $P_0 \in \mathscr{R}''$, we see that $P_0 \mathscr{R}' P_0$ is homomorphic to $\mathscr{R}'$; in fact, even isomorphic, since P_0 has in its range the cyclic vector of $\pi_\omega(\mathscr{A})$, which is separating for $\mathscr{A}'$. In fact $P_0 \pi_\omega(\mathscr{A})'' P_0 = P_0 \mathscr{R}'' P_0$ is maximal Abelian (in $P_0 \mathscr{H}$), since if it were not, $|\Omega\rangle = P_0 |\Omega\rangle$ could not be cyclic. Thus, $P_0 \mathscr{R}'' P_0 = P_0 \mathscr{R}' P_0$ and the Abelianness of $P_0 \mathscr{R}'' P_0$ implies the same for $\mathscr{A}' \cap U'$.

(i) $\Rightarrow$ (vi): Because of (vii) we know

$$\begin{aligned} \mathscr{R}' P_0 = P_0 \mathscr{R}' P_0 \subset P_0 \mathscr{R}'' P_0 = P_0 \mathscr{A}'' P_0 = P_0 \eta(\pi_\omega(\mathscr{A})'')\, P_0 \\ \subset P_0 \mathscr{Z}_\omega \cap U' P_0 \subset \mathscr{Z}_\omega P_0 \end{aligned}$$

Thus $\forall a' \in \mathscr{A}' \cap U'$, $\exists z \in \mathscr{Z}_\omega \cap U'$ such that $a' |\Omega\rangle = z |\Omega\rangle$. Since Ω is separating for $\mathscr{A}'$, we conclude that $\mathscr{A}' \cap U' \subset \mathscr{Z}_\omega$.

(vi) $\Rightarrow$ (vii): This is obvious. ∎

Definition 3. A C^*-dynamical system $(\mathscr{A}, \alpha)$ is called *η-Abelian* iff (i) is satisfied, i.e.,

$$[\eta(a), b] = 0, \qquad \forall a, b \in \pi_\omega(\mathscr{A})$$

in all extremal invariant states ω. If $\lim_{n \to \infty} [\alpha^n(a), b]$ exists as a weak (resp. strong) limit for all $a, b \in \pi_\omega(\mathscr{A})$ and

$$\lim_{n \to \infty} [\alpha^n(a), b] = 0$$

holds, the system is called *weak* (resp. *strong*) *asymptotic Abelian.*

Remark. The definition says: strong asymptotic Abelian $\Rightarrow$ weak asymptotic Abelian $\Rightarrow$ η-Abelian $\Rightarrow$ (i)–(vii) of the preceding proposition.

3. ERGODIC PROPERTIES

Next we give a precise meaning to the intuitive notion that the algebra $\mathscr{A}$ is being mixed by α. There are various candidates for a good definition.

Proposition 2. Among the properties a C^*-dynamical system might possess,

(i) $\mathscr{I} = c\mathbb{1}$, $c \in \mathbb{C}$

(ii) $\forall 0 < a, b \in \mathscr{A}$, $\exists n$: $a\alpha^n(b)\, a > 0$

(iii) there is no α- and $*$-invariant closed (linear) proper subspace (other than $\mathbb{C}\mathbb{1}$ and 0) of the Banach space $\mathscr{A}$

(iv) the linear span of $\alpha^n(a)$ and $\alpha^n(a^*)$, $n \in \mathbb{Z}$, is dense in $\mathscr{A}$ for all $a \in \mathscr{A}$ $(a \neq c\mathbb{1})$

(v) there is no α- and $*$-invariant proper nontrivial closed subalgebra of $\mathscr{A}$

(vi) the algebra generated by $\alpha^n(a)$ and $\alpha^n(a^*)$ is dense in $\mathscr{A}$ for all $a \in \mathscr{A}$ $(a \neq c\mathbb{1})$

we have the following implications:

$$\begin{array}{ccc} \text{(i)} & \Leftarrow & \text{(ii)} \\ \Uparrow & & \Uparrow \\ \text{(v)} & \Leftarrow & \text{(iii)} \\ \Updownarrow & & \Updownarrow \\ \text{(vi)} & \Leftarrow & \text{(iv)} \end{array}$$

Proof. (ii) $\Rightarrow$ (i): Suppose $\exists 0 < b \in \mathscr{I}$, $b \neq c\mathbb{1}$. So b has at least two different spectral values $c_{1,2}$. With Θ denoting the step function and $c = \frac{1}{2}(c_1 + c_2)$, we construct $b_\pm = [\pm(b - c\mathbb{1})]\, \Theta[\pm(b - c\mathbb{1})] > 0$, which are in $\mathscr{I}_+ > 0$ and $b_+ b_- = 0$.

(iii) $\Rightarrow$ (ii): $\mathscr{F}_a = \{b\colon a\alpha^n(b)\, a = 0, \forall n \in \mathbb{Z}\}$ is a linear invariant subspace. Thus, if (ii) fails for some a and b, then there exists a nontrivial $\mathscr{F}_a$.

(iii) $\Rightarrow$ (iv): If for some $a \in \mathscr{A}$ the linear span of $\alpha^n(a)$ and $\alpha^n(a^*)$ is not dense in $\mathscr{A}$, then (iii) fails.

(iv) $\Rightarrow$ (iii): If $\mathscr{B}$ is the subspace for which (iii) fails, for any $b \in \mathscr{B}$, the span of $\alpha^n(b) \subset \mathscr{B}$ and thus (iv) fails, too.

(v) $\Rightarrow$ (i): If $\mathscr{I} \neq c\mathbb{1}$, there is a proper invariant subalgebra ($\mathscr{I} = \mathscr{A}$ is impossible, since $\alpha \neq \mathrm{id}$).

(vi) $\Rightarrow$ (v): The algebra generated by $\alpha^n(a)$ and $\alpha^n(a^*)$ is obviously α- and $*$-invariant.

(v) $\Rightarrow$ (vi): The algebra generated by $\alpha^n(a)$ and $\alpha^n(a^*)$ for a any element of a proper invariant subspace cannot be dense in $\mathscr{A}$.

(iii) $\Rightarrow$ (v) and (iv) $\Rightarrow$ (vi): These are obvious. ▮

Remarks

1. Property (i) is too weak and does not imply (ii). Consider the one-dimensional shift on the algebra of $\mathscr{C}_0(\mathbb{R}^2) \cup \mathbb{I}$. Property (ii) obviously fails, but the invariant functions are proportional to $\mathbb{I}$.

2. Property (iv) requires too much. For instance, free fields in one dimension with α the shift satisfy (ii), but the linear space generated by the creation and annihilation operators is far from being all of $\mathscr{A}$.

3. Property (i) is already strong enough to exclude finite quantum systems, in fact all inner automorphisms. If $\alpha(a) = U^{-1}aU \neq a$, then $U \in \mathscr{I}$, $U \neq c\mathbb{I}$.

Definition 4. A C^*-dynamical system $(\mathscr{A}, \alpha)$ is called *transitive* (resp. *transitive mixing*) iff

$$\forall 0 < a, b \in \mathscr{A}\ \exists n \in \mathbb{Z}\colon a\alpha^n(b)\, a > 0$$

[resp.

$$\forall 0 < a, b \in \mathscr{A}\ \exists N \in \mathbb{Z}\colon a\alpha^n(b)\, a > 0\ \forall n > N]$$

The Abelian Case

To connect classical dynamical systems and their topological theory with these notions, we have the following results.

Proposition 3. For a seperable Abelian C^*-dynamical system, being transitive is equivalent to each of the following conditions:

(i) $\exists m_0 \in M\colon \bigcup_{n \in \mathbb{Z}} \alpha_*^n(m_0)$ is dense in M

(ii) $\forall N \subset M$ with $\alpha_*(N) = N$ and N closed $\Rightarrow N = M$ or N is nowhere dense in M

(iii) $\forall N \subset M$ with $\alpha_*(N) = N$ and N open $\Rightarrow N = \varnothing$ or N is dense in M

(iv) $\forall N_1, N_2 \subset M$ open and nonempty $\exists n \in \mathbb{Z}$ with $\alpha_*^n(N_1) \cap N_2 \neq \varnothing$

(v) $\forall 0 < f, g \in C(M)\ \exists n \in \mathbb{Z}$ with $g\alpha^n(f) > 0$

Proof. (i) $\Rightarrow$ (ii): Suppose there is an open, nonempty $O \subset N$. Then there exists an i with $\alpha_*^i(m_0) \in O \subset N$ and, as N is invariant under α_* it follows that also $\bigcup_{n \in \mathbb{Z}} \alpha_*^n(m_0) \subset N$. Therefore, either N has no interior or $N = M$.

(ii) $\Leftrightarrow$ (iii): This is obvious.

(iii) $\Rightarrow$ (iv): $\bigcup_{n \in \mathbb{Z}} \alpha_*^n(N_1)$ is an open α_*-invariant subset, so it must be dense in M by (iii). Therefore it has a nonempty intersection with any open subset N_2. Thus, (iv) must hold for at least one $n \in \mathbb{Z}$.

(iv) $\Rightarrow$ (i): As the C^*-algebra is separable, there exists a countable (topological) base $\{U_i\}$ of M (ref. 9, p. 17). Then the set of all points in M with nondense orbit is $\bigcup_{i \in \mathbb{Z}} \bigcap_{n \in \mathbb{Z}} \alpha_*^n(M \backslash U_i)$, since for every such point there must be a U_i which its orbit does not reach. The complementary set, which consists of all points with dense orbit, is $\bigcap_{i \in \mathbb{Z}} \bigcup_{n \in \mathbb{Z}} \alpha_*^n(U_i) \neq \varnothing$; because of (iv), it is the countable intersection of dense open sets (Baire category theory).

(iv) $\Rightarrow$ (v): There are nonempty open sets N_f and N_g for which f resp. g are strict positive for all points in N_f resp. N_g. As $\exists n \in \mathbb{Z}$ with $\alpha_*^{-n}(N_f) \cap N_g \neq \varnothing$, clearly all values of $\alpha^n(f)\, g$ in this set are strict positive.

(v) $\Rightarrow$ (iv): For all nonempty open sets N_1 and N_2 there are some functions $f_1, f_2 \in C(M)$ which are strict positive on them and vanish outside. As $\alpha^n(f_1)\, f_2 > 0$ for some n, there must exist a nonempty open subset $N \subset M$ with $\alpha^n(f_1)\, f_2$ strict positive on N. As $N \subset a^{-n}(N_1)$ and $N \subset N_2$, it follows that $\varnothing \neq N \subset \alpha(N_1) \cap N_2$.

The equivalence of transitivity and (v) is obvious. ∎

Remarks

1. Conditions like (iv) and (v) hold for (topologically) mixing Abelian systems, too.

2. For transitivity it is not necessary for every orbit to be dense; isolated fixed points can be ignored.

3. A simple example of a Hamiltonian system restricted to an energy shell where the absence of (continuous) nonconstant invariant functions is not sufficient for transitivity is given by the double-well potential:

$$V(x) = E_0 - c_1 x^2 + c_2 x^4, \qquad c_i \geqslant 0$$

For $H = E_0$ we have three orbits, one $x = 0$, one with $x > 0$, and one $x < 0$. None is dense, so the system cannot be transitive. However, a time-invariant (continuous) function must have constant values γ_1 for $x > 0$ and γ_2 for $x < 0$. Continuity of the invariant function requires $\gamma_1 = \gamma_2$. Nevertheless, this system restricted to $x \geqslant 0$ would be transitive, though there would be still a fixed point at $x = 0$.

Definition 5. An invariant state ω is called *ergodic* (resp. *mixing*) iff

$$\eta(\omega(a\alpha^n(b)\, c)) = \omega(ac)\, \omega(b)$$

for all a, b, $c \in \mathscr{A}$ [resp.

$$\lim_{n \to \infty} \omega(a\alpha^n(b)\, c) = \omega(ac)\, \omega(b)$$

for all a, b, $c \in \mathscr{A}$].

Remarks

1. The ergodic states are a subset of the extremal invariant states. Convex combination of ergodic states cannot be ergodic.

2. In the Abelian (or η-Abelian) situation the definition of ergodicity reduces to the usual

$$\eta(\omega(a\alpha^n(b))) = \omega(a)\, \omega(b)$$

(similarly for mixing).

In the noncommuting case we need three factors for a definition that excludes finite quantum systems. For them the extremal invariant states are expectation values with eigenvectors of the Hamiltonian.

If $\alpha(a) = e^{iH} a e^{-iH}$, $H\,|k\rangle = e_k\,|k\rangle$, and $\omega(a) = \langle k|\, a\,|k\rangle$, then

$$\eta_n(\omega(a\alpha^n(b))) = \eta_n\left(\sum_j \langle k|\, a\,|j\rangle\langle j|\, b\,|k\rangle\, e^{in(e_j - e_k)}\right) = \langle k|\, a\,|k\rangle\langle k|\, b\,|k\rangle$$

iff (e_k) is nondegenerate. However, there are always a, b, c such that $\eta(\omega(a\alpha^n(b)\, c)) \neq \omega(ac)\, \omega(b)$.

3. More generally, the more stringent condition in the preceding definition can never be met if α is an inner automorphisms $\alpha(a) = uau^{-1}$. Taking b as this u, we see that it would have to be a multiple of $\mathbb{1}$.

4. Note that if all invariant states $\omega \in \mathscr{S}_\alpha$ are mixing, this implies asymptotic Abelianness, since $\omega(a\alpha^n(b)\, cd) \to \omega(acd)\, \omega(b) \leftarrow \omega(ac\alpha^n(b)\, d)$.

The Abelian Case

In this case our definition reduces to the classical one and we have the following results.

Proposition 4. For an Abelian W^*-dynamical system, its measure μ being ergodic is equivalent to each of the following conditions:

(i) $\forall f$ μ-measurable

$$\eta(f(m_0)) := \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\alpha_*^i(m_0)) = \int_M f(m)\, \mu(dm) =: \mu(f)$$

for μ-almost all $m_0 \in M$

(ii) $\forall N \subset M$ with $\mu(N) > 0 \Rightarrow \mu(\bigcup_{n \in \mathbb{Z}} \alpha_*^n(N)) = 1$

(iii) $\forall N \subset M$ μ-measurable and with $\alpha_*(N) = N \Rightarrow \mu(N) = 1$ or $\mu(N) = 0$

(iv) $\forall N_1, N_2 \subset M$ with $\mu(N_1) > 0 < \mu(N_2)$ $\exists n \in \mathbb{Z}$: $\mu(\alpha_*^n(N_1) \cap N_2) > 0$

(v) $\forall f, g$ μ-measurable $\eta(\mu(fa^n(g))) = \mu(f)\, \mu(g)$

Proof. As the proofs of (i) and (v) are rather complicated and technical—for example (i) relies heavily on some classical ergodic theorems (such as the Birkhoff ergodic theorem, etc.)—and those for (ii)–(iv) are quite trivial (as with transitivity), we omit them and refer to ref. 9 [p. 34 for (i), p. 27 for (ii)–(iv), and p. 45 for (v)] or ref. 11. ∎

Remarks

1. Again similar conditions hold for mixing Abelian W^*-dynamical systems.

2. Condition (i) is one of the main results of classical ergodic theory. It states that for classical ergodic systems the *time mean* and the *phase mean* of some observable are equal almost everywhere.

3. Condition (ii) somehow corresponds to the first criterion of transitivity: The orbit of some (measurable) subset of the space will travel around almost everywhere. However, of the two notions ergodicity and transitivity, generally neither property implies the other.

4. For a transitive W^*-dynamical system any normal invariant state ω is ergodic. The reason is simply that $\eta(a)$ belongs to the algebra, is invariant, and, according to (10), a multiple of unity. Therefore, in this case $\omega(\eta(a)\, b) = \omega(a)\, \omega(b)$ and the time-invariant state is unique (*unique ergodicity*[11]).

Of course, transitivity of $(\mathscr{A}, \alpha)$ does not imply it for $(\pi_\omega(\mathscr{A})'', \alpha)$. In particular, in the Abelian situation the mean of continuous functions, the existence of which is asserted by the Birkhoff ergodic theorem (almost everywhere), will not be a continuous function any more.

4. RELATIONS

Proposition 5. If all extremal invariant states of a C^*-dynamical system $(\mathscr{A}, \alpha)$ are ergodic (resp. mixing), then $(\mathscr{A}, \alpha)$ is η-Abelian (resp. weak asymptotic Abelian).

Conversely, if $(\mathscr{A}, \alpha)$ is η-Abelian, all extremal invariant states are ergodic.

Proof. By definition,

$$\eta(\omega(c^*\alpha^n(a)\, bc)) = \omega(c^*bc)\, \omega(a) = \eta(\omega(c^*b\alpha^n(a)\, c))$$

(resp. the same for $\lim_{n\to\infty}$). Thus ergodicity (resp. mixing) implies η-Abelianness (resp. weak asymptotic Abelianness).

Conversely, if ω is extremal invariant, the invariant elements in the commutant must be proportional to $\mathbb{1}$; otherwise one could decompose ω further. η-Abelianness implies that $\eta(a)$ is in $\mathscr{A}' \cap U'$ and is thus proportional to $\mathbb{1}$. Therefore, $\eta(a) = \omega(a)\, \mathbb{1}$ and

$$\eta(\omega(a\alpha^n(b)\, c)) = \omega(\alpha\eta(b)\, c) = \omega(ac)\, \omega(b). \quad \blacksquare$$

η-Abelianness implies many ergodic properties for extremal invariant states (cf. [8, p. 149] or [3, 2]).

Proposition 6. If $(\mathscr{A}, \alpha)$ is an η-Abelian C^*-dynamical system, the following conditions are equivalent:

(i) ω is extremal invariant
(ii) ω is ergodic
(iii) $\eta(a) = c\mathbb{1}, \forall a \in \mathscr{A}$
(iv) $\pi_\omega(\mathscr{A})' \cap U' = c\mathbb{1}$
(v) $\dim P_0 = 1$
(vi) ω is the only normal invariant state on $\pi_\omega(\mathscr{A})''$

Furthermore, the following sharpened versions are equivalent:

(i) ω is mixing
(ii) $w-\lim_{n\to\pm\infty} U^n = |\Omega\rangle\langle\Omega|$
(iii) $w-\lim_{n\to\pm\infty} \alpha^n(a) = \omega(a)\, \mathbb{1}$.

Remark. In these cases η (resp. $\lim_{n\to\infty}$) maps $\mathscr{A}$ into $\mathscr{A}''$, but still these limits can be only weak limits, in general

$$\omega(ab) = \lim_{n\to\infty} \alpha^n(ab) \neq \lim_{n\to\infty} \alpha^n(a) \lim_{n\to\infty} \alpha^n(b) = \omega(a)\, \omega(b)$$

Sufficient Conditions for Transitivity

All the following conditions imply that $(\mathscr{A}, \alpha)$ is transitive:

(i) There exists a faithful ergodic state

(ii) $(\mathscr{A}, \alpha)$ is strong asymptotic Abelian and $\mathscr{A}$ is simple

(iii) $(\mathscr{A}, \alpha)$ is η-Abelian and $\mathscr{A}$ is a UHF-algebra

(iv) There is an extremal invariant state which remains faithful when extended to $\mathscr{A}''$

Proof. (i) Ergodicity states that $\eta(\omega(a\alpha^n(b)\,a)) = \omega(a^2)\,\omega(b)$. So for ω faithful and any $0 < a, b \in \mathscr{A}$ the right-hand side is always >0.

On the other hand, if there is no $n \in \mathbb{Z}$ for which $a\alpha^n(b)\,a > 0$, then for $a\alpha^n(b)\,a \in \mathscr{A}_+$ the mean must be zero, as in this case $\omega(a\alpha^n(b)\,a) = 0$ for all $n \in \mathbb{Z}$.

(ii) First consider $\eta(\omega(a^*\alpha^n(b^*)\,c^*\alpha^n(d)\,c\alpha^n(b)\,a))$ for $d > 0$, $a, b, c \in \mathscr{A}$, and ω extremal invariant. This is equal to

$$\begin{aligned}&\eta(\omega(a^*c^*\alpha^n(b^*)\,\alpha^n(d)\,\alpha^n(b)\,ca))\\ &\quad = \omega(a^*c^*ca)\,\omega(b^*db) \qquad \text{as} \quad [c, \alpha^n(b)] \to 0\end{aligned}$$

and, since $(\mathscr{A}, \alpha)$ is *a fortiori* η-Abelian, ω is ergodic.

Now assume $c^*\alpha^n(d)\,c = 0\ \forall n$. Then $\omega(b^*\,db)$ or $\omega(a^*c^*ca)$ must be zero for all a, b. Thus, $\pi_\omega(d)$ or $\pi_\omega(c^*c)$ has to be zero, which is impossible, since a simple algebra has only faithful representations.

(iii) For a UHF-algebra there is a unique trace state τ which is faithful and invariant for all automorphisms. Furthermore, $\pi_\tau(\mathscr{A})'' \cap \pi_\tau(\mathscr{A})' = c\mathbb{1}$, since any nontrivial element of the center would generate another trace state. Thus, it has to be extremal invariant because any decomposition in invariant states uses elements from $\mathscr{A}' \cap U'$ which are trivial because of (Prop. 1). Thus, it follows that

$$\eta(\tau(c^*\alpha^n(d)\,c) = \tau(c^*c)\,\tau(d)) > 0\ \forall 0 < c, d \in \mathscr{A}$$

and thus $c^*\alpha^n(d)\,c$ cannot be zero for all n.

(iv) Extremal invariant $\Rightarrow \mathscr{A}' \cap U' = c\mathbb{1}$. Faithful $\Rightarrow \exists$conjugate isomorphism $\mathscr{A}' \leftrightarrow \mathscr{A}''$: $\mathscr{A}'' \cap U' = c\mathbb{1} \Rightarrow \eta(a) = \omega(a)\,\mathbb{1} \Rightarrow b\eta(a)\,b = \omega(a)\,b^2 \neq 0 \Rightarrow$ the system is transitive and ω is ergodic. ∎

Remarks

1. Condition (i) is more or less a classical result.[9,11]

2. Whereas ergodicity may often imply transitivity, the converse is not that easy. For classical systems in fact some criteria are known where

such an implication would hold.[11] But, as was shown, for example, in ref. 10, even under quite reasonable assumptions one can construct counterexamples. Note, however, that for these systems $(\mathscr{A}, \alpha)$ but not the extension to $\mathscr{A}''$ is transitive.

Sufficient Conditions for Transitive Mixing

Any of the following implies that $(\mathscr{A}, \alpha)$ is transitive mixing:

(i) $(\mathscr{A}, \alpha)$ is η-Abelian and there exists a faithful mixing state

(ii) $(\mathscr{A}, \alpha)$ is weak asymptotic Abelian and $\mathscr{A}$ is a UHF-algebra

(iii) $(\mathscr{A}, \alpha)$ is weak asymptotic Abelian and there exists a faithful invariant state $\omega \in \mathscr{S}_\alpha$ on $\mathscr{A}''$ with $\mathscr{Z}_\omega = c\mathbb{1}$

Proof. (i) Replace η by $\lim_{n \to \infty}$ in the preceding proof.

(ii) UHF-Algebras admit a trace state τ which is faithful invariant and leads to a trivial center. The weak cluster points of $\alpha^n(a)$ must belong to the center and therefore must be $\tau(a)\,\mathbb{1}$. Thus, the time limits exist and

$$\lim_{n \to \infty} \tau(bab) = \tau(b^2)\,\tau(a) > 0 \Rightarrow b\alpha^n(a)\,b > 0 \qquad \forall n > N$$

(iii) Same as (ii), as ω has all the properties of the trace state we needed in the proof of (ii). ∎

5. DISCUSSION

We have seen that quantum mechanics offers two gifts to ergodic theory and so we want some intuitive feeling for them.

The first is that simplicity and asymptotic Abelianness already imply transitivity. This is satisfied for the shift on a two-dimensional quantum lattice system and one might wonder why this is transitive whereas we noted that the shift on the compactly supported functions on $\mathbb{R}^2$ is not. To get this conceptually straight, one has to remember that for the algebra

$$\mathscr{A} = \bigotimes_{i \in \mathbb{Z}^2} (1, \boldsymbol{\sigma}_i)$$

a norm dense set of elements is of the form $\bigotimes_{i \in I^2} (\alpha_i + \boldsymbol{\beta}_i \boldsymbol{\sigma})$ with $\alpha_i \in \mathbb{C}$, $\beta_i \in \mathbb{C}^3$, and I^2 a finite subset of $\mathbb{Z}^2$. On its complement such a product acts as a unit operator and thus its classical analog is not a function supported on a compact subset of $\mathbb{R}^2$, but on a cylindrical subset of $\{0, 1\}^{\mathbb{Z}^2}$. Transitivity means that the automorphism mixes the system such that any two observables a and a' will eventually overlap. For example, two observables which do not overlap, i.e., $aa'=0$, are $a=\bigotimes_{i\in I} P_i^+$ and $a'=\bigotimes_{i\in I'} P_i^-$, with $P^\pm$ the projectors for spin up or down and $I\cap I'\neq\emptyset$. As we shift a, eventually the shifted I will not meet I' and $\alpha^n(a)\,a'\neq 0$. Intuitively speaking, the observables have infinite tails which have to meet once.

The second gift is that for these systems the triviality of $\mathscr{Z}=\mathscr{A}''\cap\mathscr{A}'$ strengthens transitivity to mixing. This means that for observables not only the time mean, but even the time limit exists. If the time limit exists, then the average is of course the limit and transitivity becomes mixing. Now if the sequence $\alpha^n(a)$ does not converge, it has at least weak cluster points, which for an asymptotic Abelian system must be in $\mathscr{Z}$. If $\mathscr{Z}$ is trivial, all cluster points are multiples of unity. But they cannot be different, because any time-invariant state ω gives a time-invariant neighborhood which separates multiples of unity. Suppose

$$w-\lim_{k\to\infty}\alpha^{n_k}(a)=c\mathbb{1},\qquad w-\lim_{j\to\infty}\alpha^{n_j}(a)=c'\mathbb{1}$$

This would mean that there exist some K, J such that $|\omega(\alpha^{n_k}(a)-c\mathbb{1})|<\varepsilon$, $\forall k>K$, and $|\omega(\alpha^{n_j}(a)-c'\mathbb{1})|<\varepsilon$, $\forall j>J$. Since $\omega(\alpha^m(a))=\omega(a)\,\forall m$, this implies $c=c'$. Since there is no other accumulation point than $c\mathbb{1}$ and bounded sets in $\mathscr{B}(\mathscr{H})$ are weak compact, the time limit exists. In this sense for pure quantum systems, i.e., with trivial classical part, asymptotic Abelianness already guarantees the best ergodic properties.

A feature of infinite quantum systems which seems strange to the classical intuition is the existence of faithful ergodic states. Classically they are always supported by one energy shell and are never over all of phase space. Also, for finite quantum systems the extremal invariant states are expectation values with eigenvectors of the Hamiltonian and never faithful. However, infinite quantum systems should be considered as open, since the observable finite parts are in interaction with the outside. It is just that the system is not divided into an observed part and a reservoir, but this dissection is arbitrary anyway. The state specifies the condition at infinity and represents the reservoir. In any case the openness is exhibited by the fact that the time evolution is not an inner automorphism, the Hamiltonian being not an element of the algebra, but representation dependent. This opens the possibility that the system is completely mixed by the time evolution and transitivity and faithful ergodic states become possible.

Note Added: O. Bratelli has informed us after this work was completed that an equivalent definition of transitivity has been given in R. Longo and C. Peligrad, *J. Funct. Anal.* **58**:157–174 (1984).

A seemingly stronger notion was introduced in O. Bratteli, G. A. Elliott, and D. W. Robinson *J. Math. Sos. Jpn.* **37**:115–139 (1985), which also contains our sufficient condition (iv) for transitivity, Section 4.

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